3/6/26

We're proving:

<u>Thm</u> (See Th 3.2.8 and Summary 3.2.9)

Let A be an $n \times m$ matrix. Then the following are equivalent.

(a) The cols are lin. indep.

(b) The equation (*) has only the trivial solution.

(c) $\ker(A) = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^m$.

(d) $\operatorname{rank}(A) = m$

We proved

(a) $\Rightarrow$ (b) $\Rightarrow$ (c)

$\boxed{(c) \Rightarrow (d)}$

(c) says the lin syst $A\vec{x} = \vec{0}$ has only the trivial sol'n

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ &\vdots \\ x_m &= 0 \end{aligned}$$

So there are m variables, and there are no free variables.

Row reduce A.

Hence RREF(A) looks like

$$\begin{array}{cc} & \overbrace{\qquad\qquad\qquad}^{m} \\ m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

So rank(A) = m

$\boxed{(d) \Rightarrow (a)}$

Assume rank(A) = m, where m = # cols.

So RREF(A) has to be the matrix we just saw. So $A\vec{x} = \vec{0}$ has only the trivial sol'n, i.e.

$$c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0} \implies c_1 = \dots = c_m = 0$$

By Th 3.2.7, $\vec{v}_1, \dots, \vec{v}_m$ are lin. indep.

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<u>Remark</u>/<u>theorem</u> (Th. 3.2.10)

Let V be a subspace of $\mathbb{R}^p$ and let $\vec{v}_1, \dots, \vec{v}_n$ be a basis.

Let $\vec{v}$ be any element of V.

We know (since a basis spans) there are real numbers $a_1, \dots, a_n$ s.t.

$$a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{v}.$$

Assertion of this theorem: there is <u>only</u> <u>one</u> possibility for $a_1, \dots, a_n$.

proof

Say $b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{v}$ where not all $a_i = b_i$.

then $(a_1-b_1)\vec{v}_1 + \cdots + (a_n-b_n)\vec{v}_n = \vec{0}$ where not all $a_i - b_i = 0$

But this contradicts linear independence (which holds because $\vec{v}_1, \ldots, \vec{v}_n$ form a basis). This is because all $a_i - b_i$ do

$$\text{Lin indep} \Rightarrow a_1 = b_1, \ldots, a_n = b_n$$

So each element of V can be written in a unique way as a lin. comb. of basis elements!

Corollary In $\mathbb{R}^n$ the biggest linearly independent set we can find has size n, and such a set exists.

proof

Clearly $\vec{e}_1, \ldots, \vec{e}_n$ is a lin indp set, so 2nd statement is true.

Suppose you had a bigger lin. indp set $\vec{v}_1, \ldots, \vec{v}_{n+1}$. Then

$$A = \left[\vec{v}_1 \mid \cdots \mid \vec{v}_{n+1}\right]$$ is an $n \times (n+1)$ matrix.

Then $A\vec{x} = \vec{0}$ is a lin syst with $n+1$ variables and n equations, so there's a nontrivial solution, hence $\vec{v}_1, \ldots, \vec{v}_{n+1}$ aren't lin. indep //

Note the Remark/Theorem is about a subspace of $\mathbb{R}^p$ (which could be all of $\mathbb{R}^p$ but doesn't need to be) but the Corollary is (for now) about $\mathbb{R}^n$. Let's generalize it. We'll start with that next time.

Theorem (3.3.1)

Let V be a subspace of $\mathbb{R}^n$. Let $\vec{v}_1, \dots, \vec{v}_p \in V$ be lin. indep and $\vec{w}_1, \dots, \vec{w}_q$ span V. Then $p \leq q$

Should be intuitively clear